St George Girls High School

Trial Higher School Certificate Examination

2004

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Write your Student Number on every page
- All questions may be attempted.
- Begin each question on a new page
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

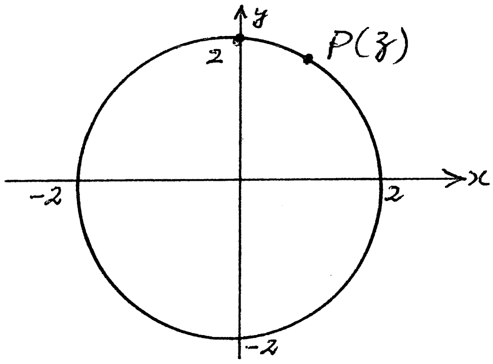
Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (15 marks) – Start a new page **Marks**

a) Evaluate $\int_0^1 \frac{dx}{(x+2)\sqrt{x+2}}$ 2

b) Find $\int \frac{x^2+5x-4}{(x-1)(x^2+1)}\,dx$ 4

c) The point P below represents the complex number z.



(i) Copy the diagram onto your answer booklet.

(ii) By considering both modulus and argument, carefully indicate on your diagram the positions of

a. Q representing $\frac{1}{z}$ 1

b. R representing $\sqrt{z}$ 1

c. S representing $z-1$ 1

d) The locus of all points z in the complex plane which satisfy $\arg\left(\frac{z-3}{z+1}\right)=\frac{\pi}{3}$ forms part of a circle.

(i) sketch this locus. 2

(ii) find the centre and radius of the circle. 2

e) Find $\sqrt{i}$ in the form $a+ib$ where a, b are real numbers. 2

Question 2 – (15 marks) – Start a new page **Marks**

a) Find

(i) $\displaystyle\int \frac{dx}{x^2+2x+5}$ 1

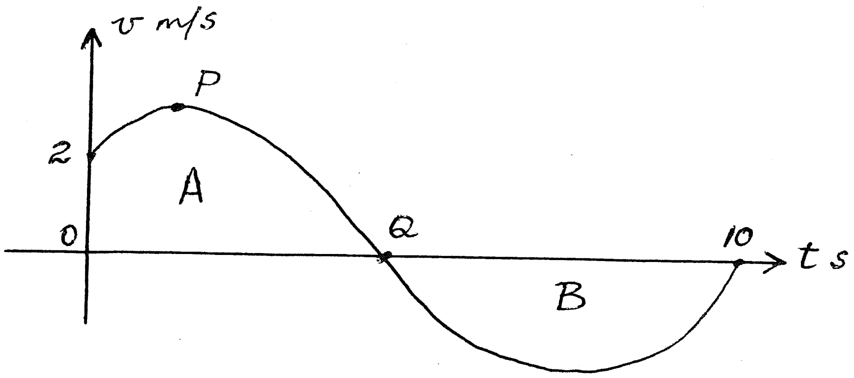
(ii) $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$ (using the substitution $x = \sec\theta$) 2

b) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos\theta}\, d\theta$ correct to 3 significant figures. 3

c) With the aid of appropriate sketches or otherwise, find the set of values of x for which the limiting sum of the following series exists.

$$1+\left(\frac{2x-3}{x+1}\right)+\left(\frac{2x-3}{x+1}\right)^2+\left(\frac{2x-3}{x+1}\right)^3+\ldots$$ 4

d) A particle started at the origin with velocity of $2m/s$. Its velocity at any time $ts\,(0 \le t \le 10)$ is shown below. Area A is 10 and Area B is 15.



(i) Identify two properties of the particle's motion at:

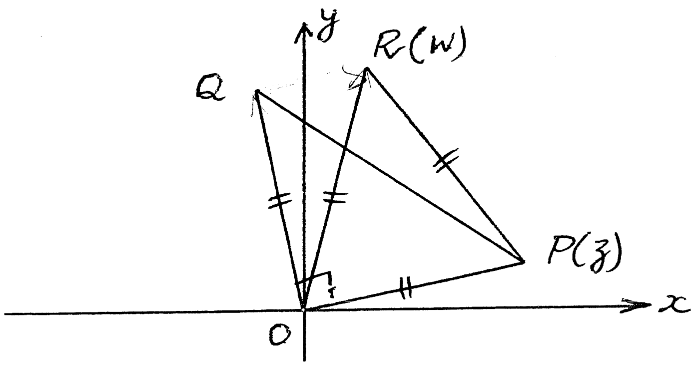
a. the maximum turning point P. 2

b. the point of inflexion Q. 2

(ii) Where is the particle at $t = 10s$? 1

Question 3 – (15 marks) – Start a new page **Marks**

a)



The point P in the complex plane above represents the complex number z. The right angled triangle OPQ is isosceles and the triangle OPR is equilateral.

(i) Find, in terms of z, the complex number represented by the point Q. 1

(ii) Find, in terms of z, the complex number which would represent the vector $\overrightarrow{QR}$ 2

(iii) If R represents the complex number w show that $w^3 + z^3 = 0$. 2

b) Consider the rectangular hyperbola $R : xy = c^2$

(i) Prove that the equation of the tangent to R at $P\left(ct, \dfrac{c}{t}\right)$ is $x + t^2 y = 2ct$ 2

(ii) The above tangent crosses the x-axis at M and the y-axis at N. If O is the origin, show that the area of triangle OMN is independent of P. 2

c) (i) Show that the equation of the normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $P\left(a\sec\theta, b\tan\theta\right)$ is $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2 + b^2$ 4

(ii) The line through P parallel to the y-axis meets the asymptote $y = \dfrac{bx}{a}$ at Q. The tangent at P meets the same asymptote at R. The normal at P meets the x-axis at G. Prove that $\angle RQG$ is a right angle. 2

Question 4 – (15 marks) – Start a new page **Marks**

a) The region bounded by the curves $y = x^2$, $y = (x-2)^2$ and the x-axis is rotated about the line $x = 2$. Use the method of cylindrical shells to find the volume of the solid. 5

b) Find the exact value of $\int_0^1 xe^{-x}\,dx$ 2

c) Consider the polynomial equation $P(x) = x^4 - 4x^2 + c$

Find those values of c for which $P(x) = 0$ has

(i) no real roots. 1

(ii) 4 different, real roots. 1

d) Consider the function $f(x) = \dfrac{x-1}{x}$

(i) Sketch the graph $y = f(x)$, showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes. 2

(ii) Use the graph $y = f(x)$ to sketch on separate axes the graphs.

(α) $y = |f(x)|$ 1

(β) $y = f(|x|)$ 1

(γ) $y = [f(x)]^2$ 1

(iii) If $g(x) = \dfrac{1}{f(x)}$, find $g'(x)$ in terms of $f(x)$ and $f'(x)$, and deduce that the x coordinates of the stationary points of $y = g(x)$ are the x coordinates of the stationary points of $y = f(x)$ for which $f(x)$ is non-zero. 1

Question 5 – (15 marks) – Start a new page **Marks**

a) The equation $2x^3 - 3x + 5 = 0$ has roots α, β, γ

(i) Find the polynomial equation with roots of

(α) $\alpha - 1, \beta - 1, \gamma - 1$ 2

(β) $\alpha^3, \beta^3, \gamma^3$ 2

(ii) Evaluate $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3} + \dfrac{1}{\gamma^3}$ 2

b) If $z_1 = 1 - \sqrt{3}i$ and $z_2 = \sqrt{3} + i$

(i) Express z_1 and z_2 in modulus-argument form. 1

(ii) Express $\left(\dfrac{z_2}{z_1}\right)$ in simplest form. 1

(iii) Find the least positive integer n for which $z_2{}^n$ is real and evaluate $z_2{}^n$ for this value of n. 2

c) (i) Prove the result:

$$\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$$ 2

(ii) Use this result to show that

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1 + \sin x}\,dx = 2$$ 3

Question 6 – (15 marks) – Start a new page **Marks**

a) A particle is dropped from rest a height h metres above the ground. At time t seconds its height above the ground, is given by

$$x = h + \frac{gt}{k} + \frac{ge^{-kt}}{k^2} - \frac{g}{k^2}$$

(i) Show that $\ddot{x} = g - kv$ where the velocity of the particle is v m/s. 2

(ii) What forces are acting on this particle? Explain carefully. 1

(iii) If it takes T seconds for the particle to reach half of its terminal velocity, show that $e^{kT} = 2$

b) Consider the curve C defined by $3x^2 + y^2 - 2xy - 8x - 16 = 0$

(i) Show that $\dfrac{dy}{dx} = \dfrac{3x - y - 4}{x - y}$ 2

(ii) Find the x coordinates of the points on C where the tangent is parallel to $y = 2x$ 2

c) If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx, \qquad n \geq 0$

(i) Show that $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, I_{n-2}$ for $n \geq 2$ 3

(ii) Hence, evaluate $\int_0^{\frac{\pi}{2}} x^4 \sin x \, dx$ 3

Question 7 – (15 marks) – Start a new page **Marks**

a) Find $\int \sec^3 x \, dx$ 3

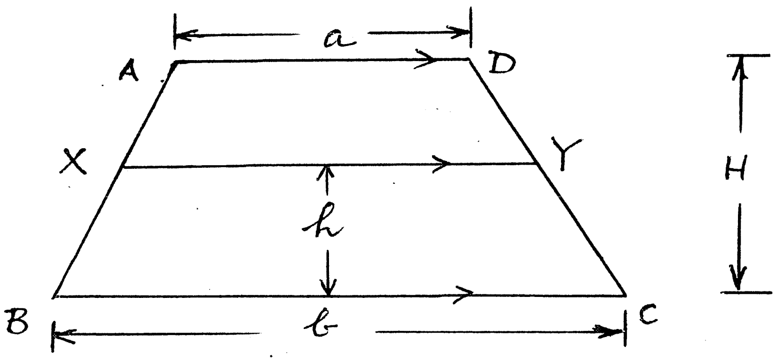
b) Sketch the region R in the Argand diagram consisting of all points z for which:

$|\arg z| \le \frac{\pi}{4}, \quad z + \bar{z} \le 4, \quad |z| \ge 2$ 3

c) Find the values of the real numbers p and q given that

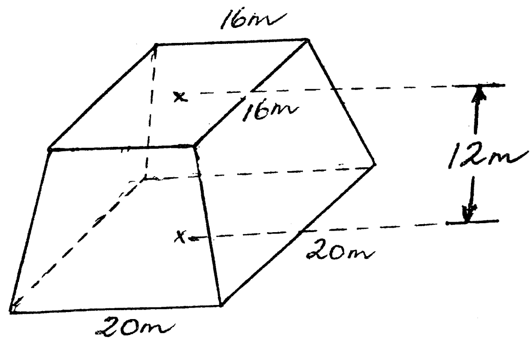
$x^3 + 2x^2 - 15x - 36 \equiv (x+p)^2(x+q)$ 3

d) (i)



$ABCD$ is an isosceles trapezium of height H with $AB = DC$. The parallel sides AD and BC are of lengths a and b respectively. X and Y are points on AB and DC respectively such that XY is parallel to BC. The perpendicular distance between XY and BC is h. Show that the length of XY is given by: $XY = b - \frac{(b-a)}{H} h$ 3

(ii)



The solid shown has a square base of $20\text{m} \times 20\text{m}$ and a square top of $16\text{m} \times 16\text{m}$.

The top and base lie on two parallel planes. The four sides are isosceles trapezia. The height of the solid is 12m. Find the volume of the solid by taking slices parallel to the base. 3

Question 8 – (15 marks) – Start a new page **Marks**

a) Find $\int \cos^4 x \, dx$ 3

b) (i) Find the complex solutions of $z^7 = 1$ 2

and hence factor $z^7 - 1$ over the real numbers. 1

(ii) Prove that $\cos\frac{3\pi}{7} + \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} = \frac{1}{2}$

c) A particle of mass 3kg moves in a straight line with velocity v *m/s* under a constant force of 5 Newtons and experiences a resistance of $2 + 3v$ N. The initial velocity of the particle was v_0 m/s.

(i) Show that the acceleration, $\ddot{x}$ m/s^2, is given by $\ddot{x} = 1 - v$ 1

(ii) Show that $v = 1 - e^{-t} + v_0 e^{-t}$ 3

(iii) Find the terminal velocity. 1

(iv) When the velocity increases from v_0 to v_1, show that the distance travelled, x m, in this time is $x = (v_0 - v_1) + \ln\left(\frac{1 - v_0}{1 - v_1}\right)$ 3

SOLUTIONS.

QUESTION 1:

(a) $$\int_0^1 \frac{dx}{(x+2)^{3/2}} = \int_0^1 (x+2)^{-\frac{3}{2}}\,dx$$

$$= \left[\frac{-2}{(x+2)^{\frac{1}{2}}}\right]_0^1$$

$$= \frac{-2}{\sqrt{3}} + \frac{2}{\sqrt{2}}$$

$$= \frac{-2\sqrt{3}}{3} + \sqrt{2}$$

$$= \frac{3\sqrt{2} - 2\sqrt{3}}{3}$$

(b) Let $\dfrac{x^2+5x-4}{(x-1)(x^2+1)} \equiv \dfrac{a}{x-1} + \dfrac{bx+c}{(x^2+1)}$

$$x^2+5x-4 = a(x^2+1) + (bx+c)(x-1)$$

$x=1$: $\quad 2 = 2a$

$\therefore a = 1$

co-eff of x^2: $\quad 1 = a+b$

$= 1+b$

$\therefore b = 0$

constant: $\quad -4 = a - c$

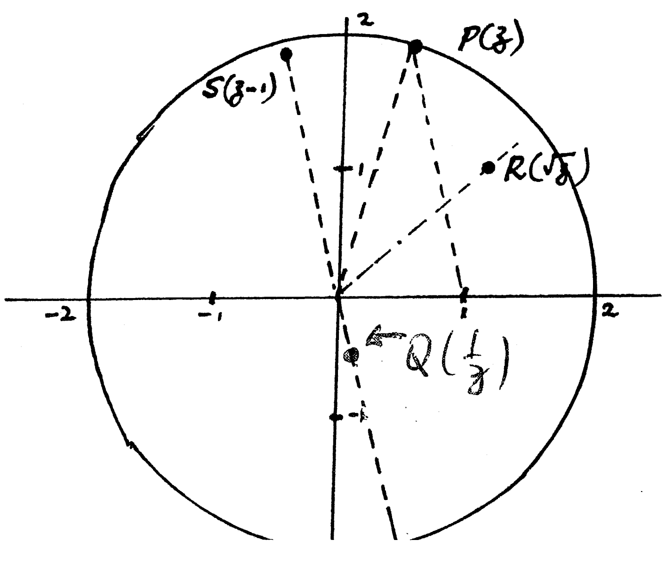
$= 1 - c$

$\therefore c = 5$

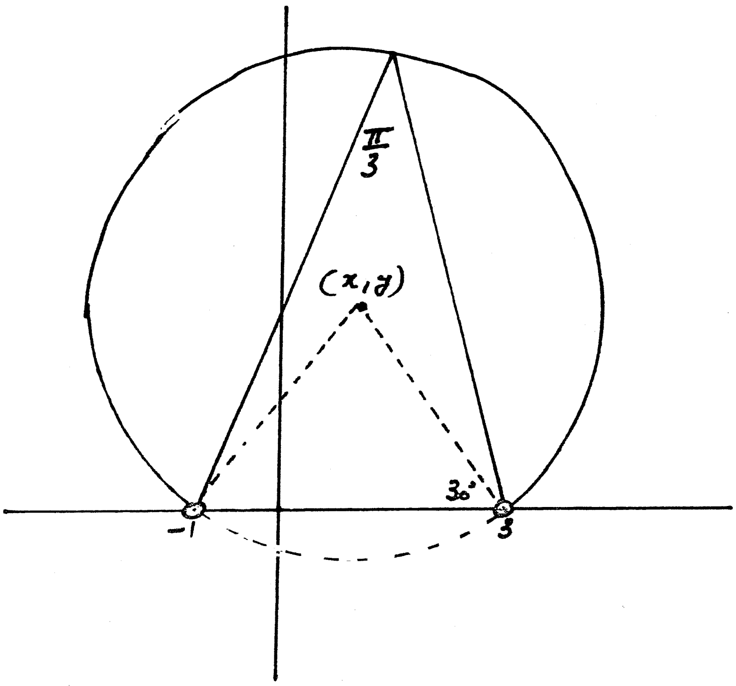
$$\therefore \int \frac{x^2+5x-4}{(x-1)(x^2+1)}\,dx = \int\left(\frac{1}{x-1} + \frac{5}{x^2+1}\right)dx$$

$$= \ln|x-1| + 5\tan^{-1}x + c$$

(c)



(d) (i)



(ii) $\tan 30° = \frac{y}{2}$

$y = 2 \tan 30°$

$x = 1$

$\therefore$ Centre is at $\left(1, \frac{2}{\sqrt{3}}\right)$

$\cos 30° = \frac{2}{r}$

$r = \frac{2}{\cos 30°}$

$= \frac{4}{\sqrt{3}}$

$\therefore$ Radius is $\frac{4}{\sqrt{3}}$

(e) $i = 1 \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right)$ k integer

$\therefore \sqrt{i} = 1 \operatorname{cis}\left(\frac{\pi}{4} + k\pi\right)$

$k=0 \Rightarrow \sqrt{i} = \operatorname{cis} \frac{\pi}{4}$

$k=1 \Rightarrow \sqrt{i} = \operatorname{cis} \frac{5\pi}{4}$

ie $\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i, \ -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

QUESTION 2 :

(a) (i) $\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$

(ii) $\int \frac{dx}{x\sqrt{x^2-1}}$ let $x = \sec\theta$
$dx = \sec\theta\tan\theta \, d\theta$

$= \int \frac{\sec\theta\tan\theta \, d\theta}{\sec\theta . \tan\theta}$

$= \theta + c$

$= \sec^{-1}x + c$

(b) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{5+3\cos\theta}$ let $t = \tan\frac{\theta}{2}$

$dt = \frac{1}{2}\sec^2\frac{\theta}{2}\, d\theta$

$= \frac{t^2+1}{2}\, d\theta$

$\therefore d\theta = \frac{2dt}{t^2+1}$

$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{5+\frac{3(1-t^2)}{1+t^2}}$

$= \int_0^1 \frac{2dt}{5+5t^2+3-3t^2}$

$= \int_0^1 \frac{2}{8+2t^2}\, dt$

$= \int_0^1 \frac{dt}{t^2+4}$

$= \frac{1}{2}\tan^{-1}\frac{t}{2}\Big]_0^1$

$= \frac{1}{2}\tan^{-1}\frac{1}{2} - 0$

$= 0.232$ (correct to 3 sig. figs)

(c) Series is geometric with $r = \frac{2x-3}{x+1}$

Limiting sum exists provided $|r| < 1$.

ie $\left|\frac{2x-3}{x+1}\right| < 1$ $x \neq -1$

ie $|2x-3| < |x+1|$

$x < -1 \Rightarrow -2x+3 < -x-1$

$-1 < x < \frac{2}{3} \Rightarrow \quad -2x+3 < x+1$

$2 < 3x$

$x > \frac{2}{3} \quad \therefore$ NO SOLUTIONS

$x > \frac{2}{3} \Rightarrow \quad 2x-3 < x+1$

$x < 4$

$\therefore$ Solution is : $\frac{2}{3} < x < 4.$

(d) (i) (a) at P we know that the particle has reached :

- maximum velocity
- zero acceleration
- a point which is less than 10 m to the right of the origin.

(b) at Q the particle has reached :

- maximum deceleration
- zero velocity
- a point which is 10 m to the right of the origin

(ii) at $t = 10$, change in displacement is -5 m

$\therefore$ Particle is 5 m to the left of the origin.

QUESTION 3:

(a) (i) $P \equiv z$

$Q \equiv z \cdot \text{cis}\frac{\pi}{2}$

$= iz$

~~(ii)~~(iii) $R \equiv z\,\text{cis}\frac{\pi}{3}$

ie $w = z\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\Rightarrow w^3 = z^3\,\text{cis}\,\pi$ by De-Moivres theorem

$= z^3(-1 + 0i)$

$= -z^3$

$\therefore w^3 + z^3 = 0$

(ii) ~~(iii)~~ $Q \equiv iz$ $\quad R \equiv z\,\text{cis}\frac{\pi}{3}$

$\therefore \vec{QR} = z\,\text{cis}\frac{\pi}{3} - iz$

$= z\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i - i\right)$

$= z\left[\frac{1}{2} + i\left(\frac{\sqrt{3}}{2} - 1\right)\right]$

(b) (i) $xy = c^2$

diff: $\Rightarrow y + x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

at $P(ct, \frac{c}{t})$ $\quad \frac{dy}{dx} = \frac{-c/t}{ct}$

$= -\frac{1}{t^2}$

$\therefore$ Tangent is $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2y - ct = -x + ct$

ie $x + t^2y = 2ct$

(ii) at M, $y = 0$ $\quad \therefore M \equiv (2ct, 0)$

at N, $x = 0$ $\quad \therefore N \equiv (0, \frac{2c}{t})$

$$\text{area } \Delta OMN = \frac{1}{2} \cdot 2ct \cdot \frac{c}{t}$$

$= c^2$ which is independent of t and hence independent of P.

(c) (i) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Differentiating $\Rightarrow$ $\dfrac{2x}{a^2} - \dfrac{2y}{b^2} \cdot y' = 0$

$$\frac{2x}{a^2} = \frac{2y}{b^2} \cdot y'$$

$$\therefore y' = \frac{b^2 x}{a^2 y}$$

at $P(a\sec\theta, b\tan\theta)$ $\quad y' = \dfrac{b^2 \cdot a\sec\theta}{a^2 \cdot b\tan\theta}$

$$= \frac{b\sec\theta}{a\tan\theta}$$

$\therefore$ Equation of normal at P is

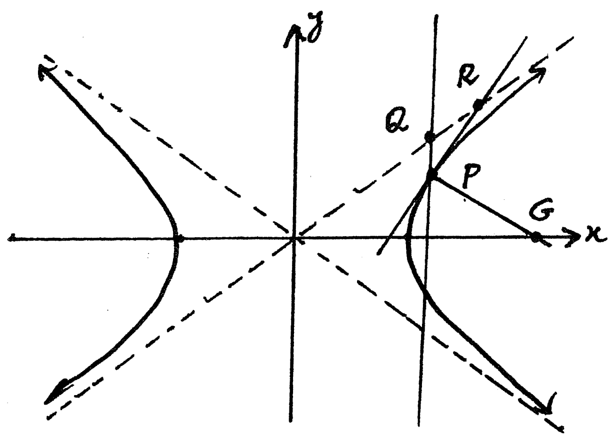
$$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta)$$

$$\therefore (b\sec\theta) \cdot y - b^2\sec\theta\tan\theta = (-a\tan\theta)x + a^2\tan\theta\sec\theta$$

ie $(a\tan\theta) \cdot x + (b\sec\theta)y = \sec\theta\tan\theta(a^2+b^2)$

$\div \sec\theta\tan\theta \Rightarrow$ $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2 + b^2$

(ii)



normal at P crosses x-axis at

$$G \equiv \left(\frac{(a^2+b^2)\sec\theta}{a}, 0\right)$$

and $Q \equiv (a\sec\theta, b\sec\theta)$

Gradient $QR = \dfrac{b}{a}$

Gradient $QG = \dfrac{b\sec\theta - 0}{a\sec\theta - \frac{a^2+b^2}{a}\sec\theta}$

$$= \frac{ab\sec\theta}{}$$

$$= \frac{ab}{-b^2}$$

$$= -\cancel{\frac{b}{a}} \quad -\frac{a}{b} .$$

$$\therefore m_{QR} \times m_{QG} = \cancel{\frac{a}{b} \times -\frac{b}{a}} \quad b/a \times -a/b .$$

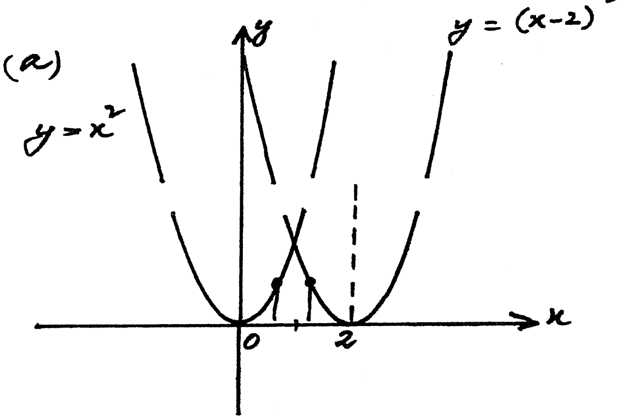
$$= -1$$

$$\therefore QR \perp QG$$

ie $R\hat{Q}G = \frac{\pi}{2}$

QUESTION 4:

(a)



Curves intersect at $(1,1)$

$\Delta V_1 = 2\pi(2-x_1)\cdot y_1\cdot\Delta x$ where $y_1 = x_1^2$ $\quad 0\le x_1\le 1$

$\Delta V_2 = 2\pi(2-x_2)\cdot y_2\cdot\Delta x$ where $y_2 = (x_2-2)^2$ $\quad 1\le x_2\le 2$

$\therefore$ Volume is given by

$$V = \lim_{\Delta x\to 0}\left[\sum_{x=0}^{1} 2\pi(2-x_1).\,x_1^2\,\Delta x + \sum_{x=1}^{2} 2\pi(2-x_2)(x_2-2)^2\,\Delta x\right]$$

$$= 2\pi\int_0^1 (2x^2-x^3)\,dx + 2\pi\int_1^2 (2-x)(x-2)^2\,dx$$

$$= 2\pi\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_0^1 - 2\pi\int_1^2 (x-2)^3\,dx$$

$$= 2\pi\left[\frac{2}{3}-\frac{1}{4}-0\right] - 2\pi\left[\frac{(x-2)^4}{4}\right]_1^2$$

$$= \frac{10\pi}{12} - 2\pi\left[0-\frac{1}{4}\right]$$

$$= \frac{4\pi}{3}$$

$\therefore$ Volume is $\frac{4\pi}{3}$ units^3.

(b) $\int_0^1 \underbrace{x}_{u}\underbrace{e^{-x}}_{dv}\,dx$

$= x.-e^{-x}\Big]_0^1 - \int_0^1 -e^{-x}\,dx$

$= \left[-1e^{-1} - 0\right] - \left[e^{-x}\right]_0^1$

$= -\frac{1}{e} - \left[\frac{1}{e} - 1\right]$

$= 1 - \frac{2}{e}$

$= \frac{e-2}{e}$

(c) $P(x) = x^4 - 4x^2 + c$

$= (x^2-2)^2 + (c-4)$

(i) no real roots $\Rightarrow$ $P(x) > 0$ for all x

and this occurs for $c - 4 > 0$

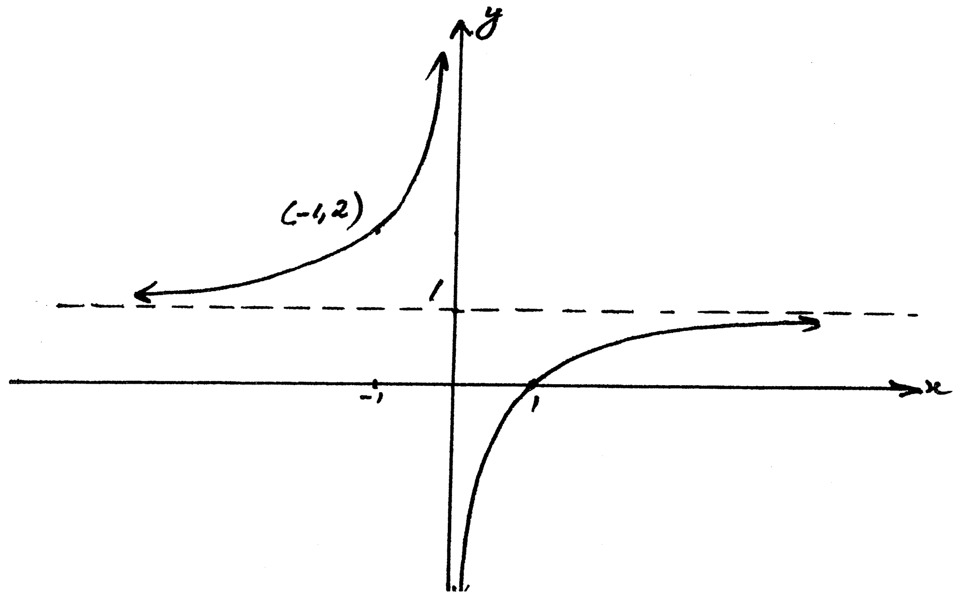
ie $c > 4$

(ii) 4 different real roots $\Rightarrow$ $c - 4 < 0$

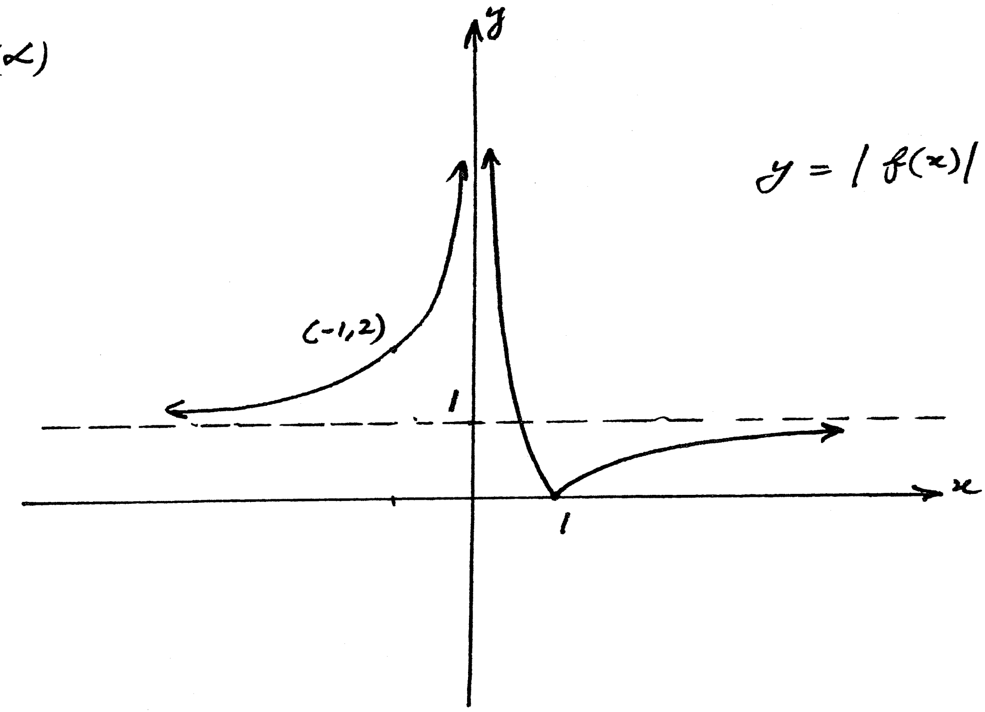
$c < 4$

(d) (i) $f(x) = \frac{x-1}{x}$

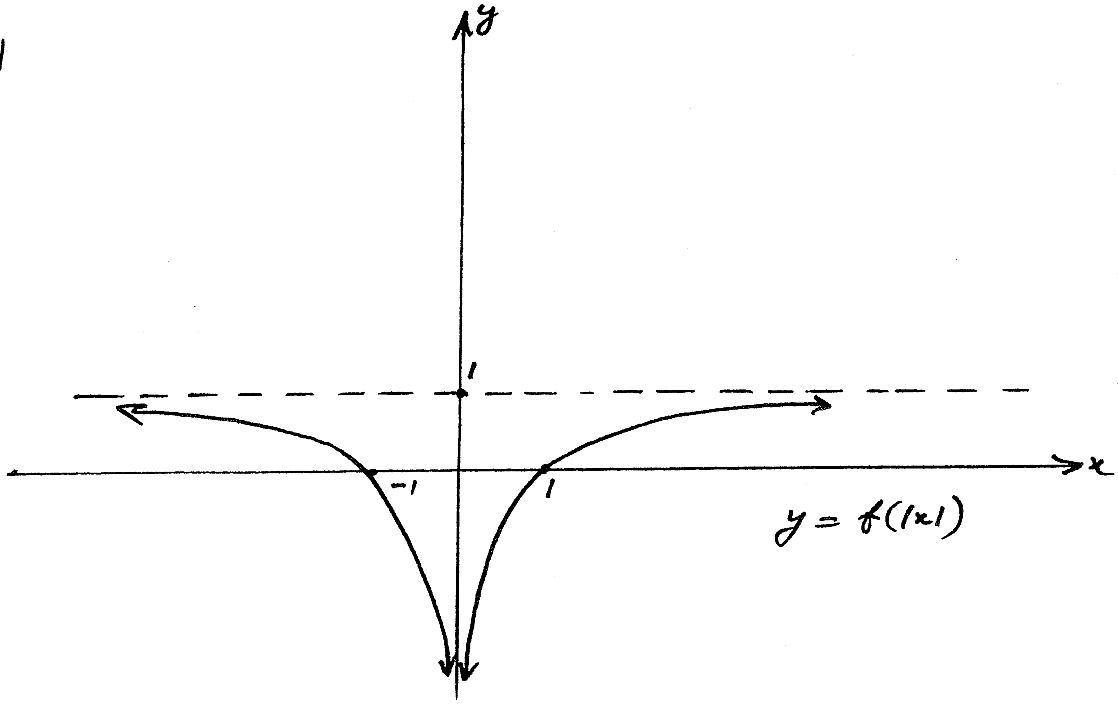
$= 1 - \frac{1}{x}$



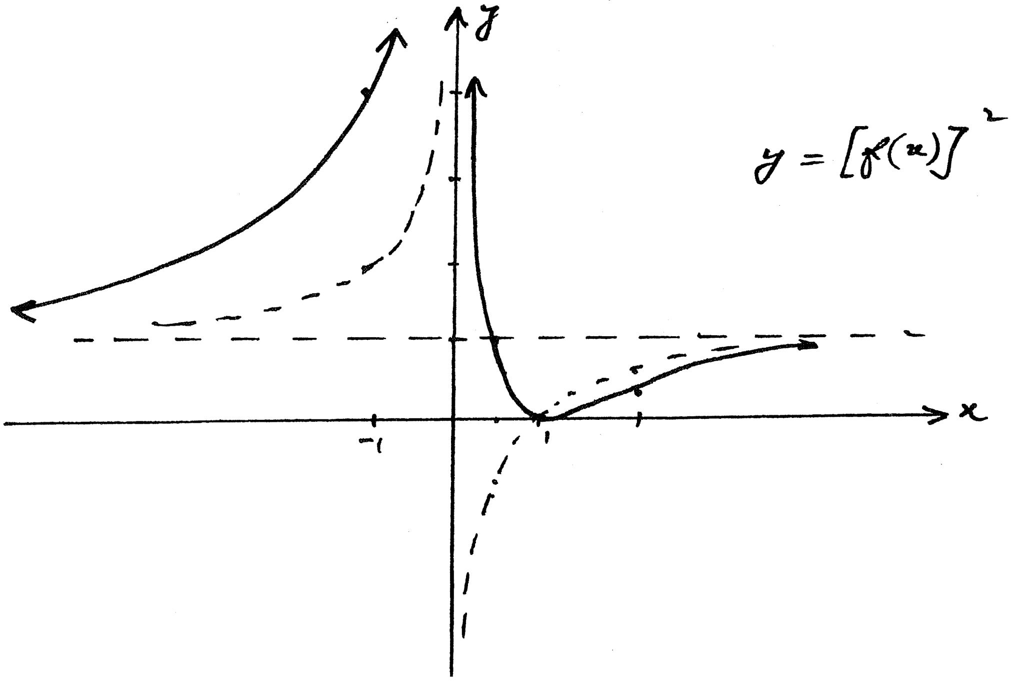
(ii) (α)



(β)



(γ)



(iii) $g(x) = \dfrac{1}{f(x)}$

$$= [f(x)]^{-1}$$

$$g'(x) = -1\,[f(x)]^{-2} \cdot f'(x)$$

$$= -\frac{f'(x)}{[f(x)]^2}$$

Hence $g'(x) = 0$ whenever $f'(x) = 0$
provided $f(x) \neq 0$.

QUESTION 5:

(a) (i) $P(x) = 2x^3 - 3x + 5$

(α) $P(x+1) = 0$

ie $2(x+1)^3 - 3(x+1) + 5 = 0$

$2x^3 + 6x^2 + 6x + 2 - 3x - 3 + 5 = 0$

$2x^3 + 6x^2 + 3x + 4 = 0$

(β) $P(\sqrt[3]{x}) = 0$

$\Rightarrow 2x - 3\sqrt[3]{x} + 5 = 0$

$3\sqrt[3]{x} = 2x + 5$

$27x = 8x^3 + 60x^2 + 150x + 125$

ie $8x^3 + 60x^2 + 123x + 125 = 0$ has roots $\alpha^3, \beta^3, \gamma^3$ — (1)

(ii) $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3} + \dfrac{1}{\gamma^3}$

$= \dfrac{\beta^3\gamma^3 + \alpha^3\gamma^3 + \alpha^3\beta^3}{\alpha^3\beta^3\gamma^3}$

$= \dfrac{123/8}{-125/8}$ from (1) above

$= -\dfrac{123}{125}$

(b) (i) $z_1 = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$ $z_2 = 2\,\text{cis}\,\frac{\pi}{6}$

(ii) $\dfrac{z_2}{z_1} = \dfrac{2\,\text{cis}\,\frac{\pi}{6}}{2\,\text{cis}\left(-\frac{\pi}{3}\right)}$

$= \text{cis}\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$

$= \text{cis}\,\frac{\pi}{2}$

$= i$

(iii) $z_2 = 2\,\text{cis}\,\frac{\pi}{6}$

$$z_2^n = 2^n \,\text{cis}\,\frac{n\pi}{6}$$

$$= 2^n\left[\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right]$$

For z_2^n to be real we need $\frac{n\pi}{6} = \pi$

ie $n = 6$

Then $z_2^6 = 64.\cos\pi$

$$= -64$$

(c) (i) $$\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{\text{let } u=-x} + \int_{0}^{a} f(x)\,dx$$

$$= \int_{a}^{0} f(-u).-du + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} \left[f(-x) + f(x)\right] dx$$

(ii) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}\,dx = \int_{0}^{\frac{\pi}{4}} \left(\frac{1}{1-\sin x} + \frac{1}{1+\sin x}\right)dx$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{2}{1-\sin^2 x}\,dx$$

$$= \int_{0}^{\frac{\pi}{4}} 2\sec^2 x\,dx$$

QUESTION 6:

(a) (i) $$x = h + \frac{gt}{k} + \frac{g}{k^2}e^{-kt} - \frac{g}{k^2} \qquad (1)$$

$$\dot{x} = \frac{g}{k} - \frac{g}{k}e^{-kt} = v \qquad (2)$$

$$\ddot{x} = ge^{-kt}$$

$$= k\left(\frac{g}{k}e^{-kt}\right)$$

$$= k\left(\frac{g}{k} - v\right) \text{ from } (2)$$

$$\ddot{x} = g - kv$$

(ii) Forces acting are gravity downwards and a resistance force upwards where downwards direction is positive.

(iii) from (2) terminal velocity is $\frac{g}{k}$

$$\therefore (2) \Rightarrow \frac{g}{k} - \frac{g}{k}e^{-kT} = \frac{g}{2k}$$

$$\frac{g}{2k} = \frac{g}{k}e^{-kT}$$

$$\text{ie } e^{-kT} = \frac{1}{2}$$

$$\therefore e^{kT} = 2$$

(b) $$C: \ 3x^2 + y^2 - 2xy - 8x - 16 = 0 \qquad (1)$$

(i) Differentiating $\Rightarrow$

$$6x + 2yy' - 2y - 2xy' - 8 = 0$$

$$y'(2y - 2x) = 8 + 2y - 6x$$

$$\therefore y' = \frac{-2(3x - y - 4)}{-2(x - y)}$$

$$= \frac{3x - y - 4}{}$$

(ii) $\parallel$ to $y = 2x \Rightarrow y' = 2$

$$\therefore \frac{3x - y - 4}{x - y} = 2$$

$$3x - y - 4 = 2x - 2y$$

$$y = 4 - x \quad \text{sub in } ①$$

$$3x^2 + 16 - 8x + x^2 - 8x + 2x^2 - 8x - 16 = 0$$

$$6x^2 - 24x = 0$$

$$6x(x-4) = 0$$

$$x = 0, 4$$

(i) $I_n = \int_0^{\frac{\pi}{2}} \underbrace{x^n}_{u} \underbrace{\sin x}_{dv}\, dx$

$$= x^n . -\cos x \Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x . n x^{n-1}\, dx$$

$$= n \int_0^{\frac{\pi}{2}} \underbrace{x^{n-1}}_{u} \underbrace{\cos x}_{dv}\, dx$$

$$= n \left[x^{n-1} . \sin x \right]_0^{\frac{\pi}{2}} - n \int_0^{\frac{\pi}{2}} \sin x . (n-1) . x^{n-2}\, dx$$

$$= n . \left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2} \qquad n \geq 2$$

(ii) $\int_0^{\frac{\pi}{4}} x^4 \sin x\, dx = I_4$

$$= 4 . \left(\frac{\pi}{2}\right)^3 - 4 . 3 . I_2$$

$$= \frac{\pi^3}{2} - 12 \left[2 . \left(\frac{\pi}{2}\right) - 2 . 1 . I_0 \right]$$

$$= \frac{\pi^3}{2} - 12\pi + 24 \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$= \frac{\pi^3}{2} - 12\pi + 24 \left[-\cos x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^3}{2} - 12\pi + 24$$

QUESTION 7:

(a) $\int \sec^3 x \, dx = \int \underbrace{\sec x}_{u} . \underbrace{\sec^2 x}_{dV} \, dx$

$= \sec x . \tan x - \int \tan x . \sec x \tan x \, dx$

$= \sec x \tan x - \int \sec x . \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$

$= \sec x \tan x - \int \sec^3 x + \int \sec x \, dx$

$\therefore 2 \int \sec^3 x \, dx = \sec x \tan x + \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx$

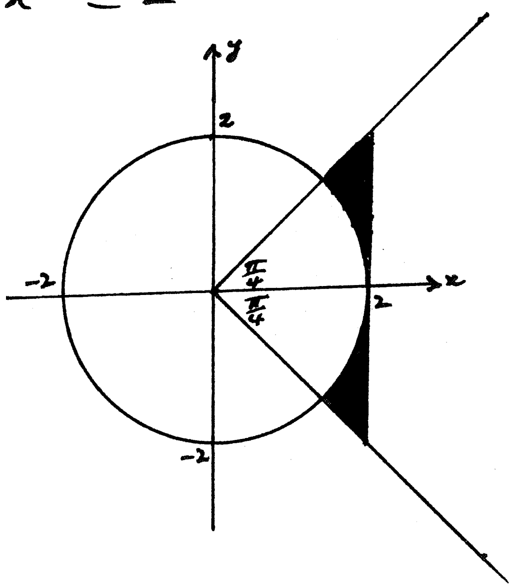
$\therefore \int \sec^3 x \, dx = \frac{1}{2} \left[\sec x \tan x + \ln |\sec x + \tan x| \right] + C$

(b) $z + \bar{z} \leq 4$

$\Rightarrow x + iy + x - iy \leq 4$

$2x \leq 4$

$x \leq 2$



(c) Let $P(x) = x^3 + 2x^2 - 15x - 36$ Repeated zero

$P'(x) = 3x^2 + 4x - 15$

$= (x+3)(3x-5)$ Repeated zero could be -3 or $\frac{5}{3}$

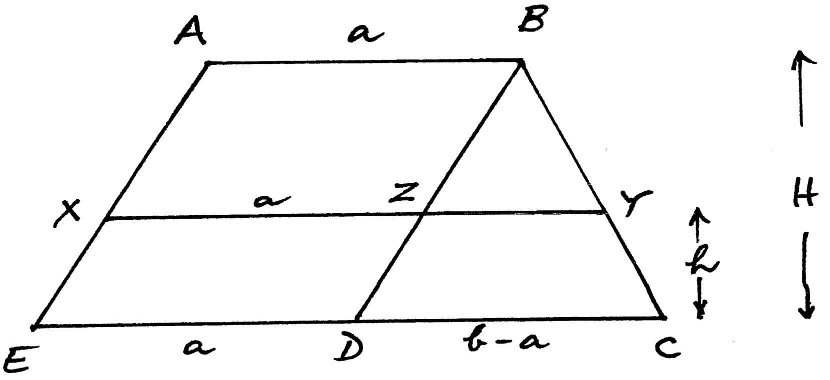
$P(-3) = -27 + 18 + 45 - 36$

$\therefore \; P(x) = (x+3)^2(x+q)$

by observation

$P(x) = (x+3)^2(x-4) \qquad \therefore p = 3, \; q = -4.$

(d) (i)



$\Delta BZY \;|||\; \Delta BDC$ (equiangular)

$$\therefore \frac{ZY}{b-a} = \frac{H-h}{H}$$

$$ZY = \frac{(b-a)(H-h)}{H}$$

$$\therefore XY = a + \frac{(b-a)(H-h)}{H}$$

$$= a + (b-a).\frac{H}{H} - \frac{(b-a)h}{H}$$

$$= b - \frac{(b-a)h}{H}$$

(ii) Take a thin slice through the solid parallel to the base and height h m above it.

Volume of slice

$$= \left[20 - \frac{(20-16).h}{12}\right]^2 . \Delta h \quad \text{from (i) above}$$

$$= \left(20 - \frac{h}{3}\right)^2 . \Delta h$$

$\therefore$ Volume of solid

$$V = \lim_{\Delta h \to 0} \sum_{h=0}^{12} \left(20 - \frac{h}{3}\right)^2 \Delta h$$

$$= \int_0^{12} \left(20 - \frac{h}{3}\right)^2 dh$$

$$= \left[\frac{\left(20 - \frac{h}{3}\right)^3}{3 \times -\frac{1}{3}} \right]_0^{12}$$

$$= -\left[16^3 - 20^3\right]$$

$$= 3904$$

$\therefore$ Volume is $3904\ m^3$

QUESTION 8:

(a) $\int \cos^4 x \, dx$ $\qquad\qquad$ $\cos 2\theta = 2\cos^2\theta - 1$

$$= \int \left(\frac{\cos 2x + 1}{2}\right)^2 dx$$

$$= \frac{1}{4}\int \left(\cos^2 2x + 2\cos 2x + 1\right) dx$$

$$= \frac{1}{4}\int \left(\frac{\cos 4x + 1}{2} + 2\cos 2x + 1\right) dx$$

$$= \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + \frac{3x}{8} + C$$

(b) (i) $z^7 = 1 = 1 \operatorname{cis}(0 + 2k\pi)$ $\qquad$ k integer

$z = \operatorname{cis}\left(\frac{2k\pi}{7}\right)$ by De Moivres theorem

$k=0 \quad z = \operatorname{cis} 0 \quad = 1$

$k=1 \quad z = \operatorname{cis} \frac{2\pi}{7} \quad = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$.

$k=2 \quad z = \operatorname{cis} \frac{4\pi}{7} \quad = -\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$ -

$k=3 \quad z = \operatorname{cis} \frac{6\pi}{7} \quad = -\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$

$k=4 \quad z = \operatorname{cis} \frac{8\pi}{7} \quad = -\cos\frac{\pi}{7} - i\sin\frac{\pi}{7}$

$k=5 \quad z = \operatorname{cis} \frac{10\pi}{7} \quad = -\cos\frac{3\pi}{7} - i\sin\frac{3\pi}{7}$ -

$k=6 \quad z = \operatorname{cis} \frac{12\pi}{7} \quad = \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$.

$$z^7 - 1 = (z-1)\left(z^2 - 2\cos\frac{2\pi}{7}z + 1\right)\left(z^2 + 2\cos\frac{3\pi}{7}z + 1\right)\left(z^2 + 2\cos\frac{\pi}{7}z + 1\right)$$

(ii) Sum of roots

$$\Rightarrow \quad 0 = 1 + 2\cos\frac{2\pi}{7} - 2\cos\frac{3\pi}{7} - 2\cos\frac{\pi}{7}$$

$$\therefore \cos\frac{3\pi}{7} + \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} = \frac{1}{2}$$

(c) (i) $m\ddot{x} = F - R$

$\therefore\ 3\ddot{x} = 5 - (2 + 3v)$

$= 3 - 3v$

$\therefore\ \ddot{x} = 1 - v$

(ii) $\dfrac{dv}{dt} = 1 - v$

$\dfrac{dt}{dv} = \dfrac{1}{1-v}$

$\therefore\ t = -\ln(1-v) + c$

at $t = 0,\ v = v_0$

$\therefore\ 0 = -\ln(1 - v_0) + c$

$c = \ln(1 - v_0)$

$\therefore\ t = \ln(1 - v_0) - \ln(1 - v)$

$= \ln\left(\dfrac{1 - v_0}{1 - v}\right)$

$\therefore\ e^t = \dfrac{1 - v_0}{1 - v}$

$\dfrac{1-v}{1-v_0} = e^{-t}$

$1 - v = e^{-t} - v_0 e^{-t}$

$\therefore\ v = 1 - e^{-t} + v_0 e^{-t}$

(iii) as $t \to \infty$ $v \to 1$

$\therefore$ Terminal velocity is 1 m/s

(iv) $v\dfrac{dv}{dx} = 1 - v$

$\dfrac{dv}{dx} = \dfrac{1-v}{v}$

$\dfrac{dx}{dv} = \dfrac{v}{1-v}$

$\displaystyle\int^{v_1} \frac{dx}{dv}\, dv = \int^{v_1} \frac{v}{1-v}\, dv$

ie $x(v_f) - x(v_0) = \int_{v_0}^{v_1} \frac{v}{1-v}\, dv$

$$= \int_{v_0}^{v_1} \left(-1 . \frac{1-v}{1-v} + \frac{1}{1-v} \right) dv$$

$$= \left[-v - \ln(1-v) \right]_{v_0}^{v_1}$$

$$= -v_1 - \ln(1-v_1) - \left[-v_0 - \ln(1-v_0) \right]$$

$$= (v_0 - v_1) + \ln\left(\frac{1-v_0}{1-v_1} \right)$$